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Simultaneous propagation of N-solitons in a fibre medium with all higher-order effects

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Abstract. We consider the N-coupled higher-order nonlinear Schrödinger (N-CHNLS) equations which govern simultaneous propagation of N fields in a fibre with all higher-order effects such as higher-order dispersion, self-steepening and stimulated inelastic scattering. Painlevé analysis is used to identify the integrable form of N-CHNLS equations. We generalize the 2×2 Ablowitz–Kaup–Newell–Segur method to a $(2N+1)\times(2N+1)$ eigenvalue problem and construct the Lax pair. Using the inverse scattering transform, one-soliton solutions are derived explicitly.

1. Introduction

The discovery of optical solitons paved the way to overcome chromatic dispersion constraints in fibre optics. Solitons in optical fibre are formed by an exact balance between group velocity dispersion (GVD) (a linear effect) and self-phase modulation (SPM) (a nonlinear effect). This was predicted theoretically by Hasegawa and Tappert [1] and observed experimentally by Mollenauer *et al* [2]. The dynamics of nonlinear wave propagation in a single-mode fibre is governed by the famous nonlinear Schrödinger (NLS) equation of the form [1, 3, 4],

$$E_Z = \mathrm{i}\left(\alpha_1 E_{TT} + \alpha_2 |E|^2 E\right) \tag{1}$$

where E is the slowly varying envelope of the axial field, α_1 and α_2 are GVD and SPM parameters, respectively, and the subscripts Z and T denote spatial and temporal partial derivatives.

For transmitting pulses at high bit rate, it is necessary to propagate ultra-short pulses. In 1986, Mitschke and Mollenauer [5] reported that ultra-short pulses suffer from a self-frequency shift due to the Raman effect. Ultra-short pulses not only suffer from stimulated Raman (or inelastic) scattering but also from higher-order dispersion (HOD) and Kerr dispersion (also called self-steepening) [3, 4, 6]. HOD is a linear effect but, unlike GVD, it broadens the pulses asymmetrically in the time domain. Kerr dispersion is due to the intensity dependence of the group velocity. This forces the peak of the pulse to travel faster than the wings, which causes an asymmetrical spectral broadening. Stimulated Raman scattering gives a self-frequency shift to pulses. The self-frequency shift is a self-induced redshift in the pulse spectrum arising from the Raman effect: long-wavelength components of the pulse experience Raman gain at the expense of short-wavelength components, resulting in an increasing redshift as the pulse propagates. It has been recognized that the self-frequency shift is a potentially detrimental effect in soliton communication systems due to the fact that the power fluctuations in the source translate into frequency fluctuations in the fibre through the power dependence of the soliton self-frequency shift and hence into timing jitter at the receiver [7]. With all of these effects, nonlinear wave propagation is governed by the higher-order nonlinear Schrödinger (HNLS) equation [3, 4, 6, 8–10] of the form,

$$E_Z = \mathrm{i}\left(\tfrac{1}{2}E_{TT} + |E|^2 E\right) - \epsilon\left[\alpha_3 E_{TTT} + \alpha_4(|E|^2 E)_T + \alpha_5 E(|E|^2)_T\right] \tag{2}$$

where α_3, α_4 and α_5 represent the HOD, self-steepening and stimulated inelastic scattering parameters, respectively.

An inverse scattering transform (IST) scheme for the HNLS equation for the condition $\alpha_3{:}\alpha_4{:}(\alpha_4 + \alpha_5) = 1{:}6{:}3$, was applied by Sasa and Satsuma [8]. Painlevé analysis and other related integrable properties of the HNLS equation were carried out in [9, 10]. With some suitable transformations and reductions, Kodama [11] reduced the HNLS equation to a Hirota equation [12].

For handling more channels it is necessary to propagate more than one field simultaneously. Transmission of many fields simultaneously in a fibre is called wavelength division multiplexing (WDM). In 1974, Manakov [13] derived the coupled NLS equations from the NLS equation by considering that the total field is comprised of two fields (left and right polarizations). In a similar way, coupled HNLS (CHNLS) equations have been proposed and it has been shown that the system equation is integrable for a particular form using Painlevé analysis [14]. The linear eigenvalue problem for CHNLS equations and exact one-soliton solutions generated using the Bäcklund transformation are given in [15]. A similar analysis was also extended to simultaneous propagation of three fields. The bilinear form for the CHNLS equations and associated soliton solutions were constructed in [16].

Here we consider simultaneous propagation of N nonlinear waves in a fibre medium with all higher-order effects namely, HOD, self-steepening and stimulated inelastic scattering. The wave dynamics of the system is governed by N-CHNLS equations of the form,

$$E_{jZ} = \mathrm{i}\left(\tfrac{1}{2}E_{jTT} + \sum_{n=1}^{N}|E_n|^2 E_j\right) - \epsilon\left[\alpha_3 E_{jTTT} + \alpha_4\left(\sum_{n=1}^{N}|E_n|^2 E_j\right)_T + \alpha_5 E_j\left(\sum_{n=1}^{N}|E_n|^2\right)_T\right]$$
$$j = 1, 2, \ldots, N. \tag{3}$$

Here in equation (3) the weighting factors between the self-phase modulation effects and the cross-phase modulation effects are considered equally. Usually they will not be equal but for the following two cases, one can consider equal weighting factors. Case (i) for a purely electrostrictive nonlinearity [17] and case (ii) in the case of elliptical bifringence [18].

In this paper we use Painlevé singularity structure analysis to identify an integrable form of the system equation (3). To derive the Lax pair for the integrable form of N-CHNLS equations, the 2×2 Ablowitz–Kaup–Newell–Segur (AKNS) formalism [19] is generalized to a $(2N + 1) \times (2N + 1)$ linear eigenvalue problem. Finally, an IST scheme [20] is used to generate exact single-soliton solutions.

2. Painlevé analysis

A new set of variables a_j $(= E_j)$ and b_j $(= E_j^*)$ $(j = 1, 2, \ldots, N)$ are introduced for the purpose of Painlevé singularity structure analysis [21]. Thus, using equation (3), a_j and b_j

can be written as

$$
\begin{aligned}
a_{jZ} &= \mathrm{i}\left(\tfrac{1}{2}a_{jTT} + \sum_{n=1}^{N} a_n b_n a_j\right) - \epsilon\left[\alpha_3 a_{jTTT} + \alpha_4\left(\sum_{n=1}^{N} a_n b_n a_j\right)_T + \alpha_5 a_j\left(\sum_{n=1}^{N} a_n b_n\right)_T\right] \\
b_{jZ} &= -\mathrm{i}\left(\tfrac{1}{2}b_{jTT} + \sum_{n=1}^{N} a_n b_n b_j\right) - \epsilon\left[\alpha_3 b_{jTTT} + \alpha_4\left(\sum_{n=1}^{N} a_n b_n b_j\right)_T + \alpha_5 b_j\left(\sum_{n=1}^{N} a_n b_n\right)_T\right].
\end{aligned}
\tag{4}
$$

Generalized Laurent series expansion of a_j and b_j are

$$
\begin{aligned}
a_j &= \phi^{\mu_j}\sum_{k=0}^{\infty} a_{jk}(Z,T)\phi^k \\
b_j &= \phi^{\delta_j}\sum_{k=0}^{\infty} b_{jk}(Z,T)\phi^k
\end{aligned}
\tag{5}
$$

with $a_{j0}, b_{j0} \neq 0$, where μ_j and δ_j are negative integers, a_{jk} and b_{jk} are a set of expansion coefficients which are analytic in the neighbourhood of the non-characteristic singular manifold $\phi(Z,T) = T + \varphi(Z) = 0$. Looking at the leading order, $a_j \approx a_{j0}\phi_j^{\mu_j}$ and $b_j \approx b_{j0}\phi_j^{\delta_j}$ are substituted in equation (4) and upon balancing dominant terms, the following results are obtained:

$$
\begin{aligned}
&\mu_j = \delta_j = -1 \\
&\sum_{n=1}^{N} a_{n0}b_{n0} = \frac{-6\alpha_3}{3\alpha_4 + 2\alpha_5}.
\end{aligned}
\tag{6}
$$

Substituting full Laurent series and considering leading-order terms alone, the resonances are found to be

$$
k = -1, \underbrace{0, 0, 0, \ldots, 0}_{(2N-1)\ \text{times}}, 3, 4 \qquad \text{and} \qquad \underbrace{3 \pm 2\sqrt{(-\alpha_5)/(3\alpha_4 + 2\alpha_5)}}_{(2N-1)\ \text{times}}. \tag{7}
$$

The resonance value at $k = -1$ represents the arbitrariness of the singularity manifold $\phi(Z,T) = T + \varphi(Z) = 0$, while resonances at $k = (2N-1)$ zeros are associated with the arbitrariness of the function a_{j0} and b_{j0} (as seen in equation (6)). Also, by collecting coefficients of different powers of φ, it is seen that equation (4) admits a sufficient number of arbitrary functions at $k = 3, 4$ and $(2N-1)(3 \pm 2\sqrt{(-\alpha_5)/(3\alpha_4 + 2\alpha_5)})$ for the condition $\alpha_3{:}\alpha_4{:}(\alpha_4 + \alpha_5) = 1{:}6{:}3$. Thus from Painlevé analysis, the integrable form of equation (3) is obtained as

$$
E_{jZ} = \mathrm{i}\left(\tfrac{1}{2}E_{jTT} + \sum_{n=1}^{N}|E_n|^2 E_j\right) - \epsilon\left[E_{jTTT} + 6\sum_{n=1}^{N}|E_n|^2 E_{jT} + 3E_j\left(\sum_{n=1}^{N}|E_n|^2\right)_T\right]. \tag{8}
$$

3. IST scheme for N-CHNLS equations

In order to analyse equation (8) it is rather convenient to introduce the following variable transformations:

$$
\begin{aligned}
u_j(x,t) &= E_j(T,Z)\exp\left[\frac{-\mathrm{i}}{6\epsilon}\left(T - \frac{Z}{18\epsilon}\right)\right] \\
t &= Z \\
x &= T - \frac{Z}{12\epsilon}.
\end{aligned}
\tag{9}
$$

Then, equation (8) reduces to N-coupled complex modified Korteweg–deVries (KdV)-type equations,

$$u_{jt} + \epsilon\left[u_{jxxx} + 6\sum_{n=1}^{N}|u_n|^2 u_{jx} + 3u_j\left(\sum_{n=1}^{N}|u_n|^2\right)_x\right] = 0. \tag{10}$$

To construct a Lax pair, we generalize the 2×2 AKNS method [19] to a $(2N+1)\times(2N+1)$ linear eigenvalue problem and obtain a Lax pair for equation (10). It should be noted that the HNLS equation considered by Sasa and Satsuma [8] admits a 3×3 Lax pair and two-coupled HNLS equations and three-coupled HNLS equations [15] admit 5×5 and 7×7 Lax pairs, respectively.

The Lax pair for N-coupled complex modified KdV equations (10) is derived as

$$\begin{aligned} &\frac{\partial\Psi}{\partial x} = U\Psi \\ &\Psi = \begin{pmatrix}\Psi_1 & \Psi_2 & \Psi_3 & \cdots & \Psi_{2N+1}\end{pmatrix}^T \end{aligned} \tag{11}$$

where

$$U = \begin{pmatrix} -\mathrm{i}\zeta & 0 & \cdots & 0 & 0 & 0 & 0 & u_N \\ 0 & -\mathrm{i}\zeta & \cdots & 0 & 0 & 0 & 0 & u_N^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\mathrm{i}\zeta & 0 & 0 & 0 & u_2 \\ 0 & 0 & \cdots & 0 & -\mathrm{i}\zeta & 0 & 0 & u_2^* \\ 0 & 0 & \cdots & 0 & 0 & -\mathrm{i}\zeta & 0 & u_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -\mathrm{i}\zeta & u_1^* \\ -u_N^* & -u_N & \cdots & -u_2^* & -u_2 & -u_1^* & -u_1 & \mathrm{i}\zeta \end{pmatrix}. \tag{12}$$

ζ is the spectral parameter and an asterisk denotes a complex conjugate. Time evolution of the eigenfunction Ψ is given by

$$\frac{\partial\Psi}{\partial t} = V\Psi \tag{13}$$

$$V = -4\mathrm{i}\epsilon\zeta^3 \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$
+\,4\epsilon\zeta^2
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_N \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_N^* \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_2 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_2^* \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_1 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_1^* \\
-u_N^* & -u_N & \cdots & -u_2^* & -u_2 & -u_1^* & -u_1 & 0
\end{pmatrix}
$$

$$
+2\mathrm{i}\epsilon\zeta
\begin{pmatrix}
|u_N|^2 & u_N^2 & \cdots & u_2^*u_N & u_2u_N & u_1^*u_N & u_1u_N & u_{Nx} \\
u_N^{*2} & |u_N|^2 & \cdots & u_2^*u_N^* & u_2u_N^* & u_1^*u_N^* & u_1u_N^* & u_{Nx}^* \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
u_2u_N^* & u_2u_N & \cdots & |u_2|^2 & u_2^2 & u_2u_1^* & u_2u_1 & u_{2x} \\
u_2^*u_N^* & u_2^*u_N & \cdots & u_2^{*2} & |u_2|^2 & u_2^*u_1^* & u_2^*u_1 & u_{2x}^* \\
u_1u_N^* & u_1u_N & \cdots & u_1u_2^* & u_1u_2 & |u_1|^2 & u_1^2 & u_{1x} \\
u_1^*u_N^* & u_1^*u_N & \cdots & u_1^*u_2^* & u_1^*u_2 & u_1^{*2} & |u_1|^2 & u_{1x}^* \\
u_{Nx}^* & u_{Nx} & \cdots & u_{2x}^* & u_{2x} & u_{1x}^* & u_{1x} & -2A
\end{pmatrix}
$$

$$
+\epsilon
\begin{pmatrix}
u_{Nx}^*u_N - u_N^*u_{Nx} & 0 & \cdots & u_{2x}^*u_N - u_2^*u_{Nx} & u_{2x}u_N - u_2u_{Nx} & u_{1x}^*u_N - u_1^*u_{Nx} & u_{1x}u_N - u_1u_{Nx} & -4Au_N - u_{Nxx} \\
0 & u_N^*u_{Nx} - u_{Nx}^*u_N & \cdots & u_{2x}^*u_N^* - u_2^*u_{Nx}^* & u_{2x}u_N^* - u_2u_{Nx}^* & u_{1x}^*u_N^* - u_1^*u_{Nx}^* & u_{1x}u_N^* - u_1u_{Nx}^* & -4Au_N^* - u_{Nxx}^* \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
u_2u_{Nx}^* - u_{2x}u_N^* & u_2u_{Nx} - u_{2x}u_N & \cdots & u_{2x}^*u_2 - u_2^*u_{2x} & 0 & u_2u_{1x}^* - u_{2x}u_1^* & u_2u_{1x} - u_{2x}u_1 & -4Au_2 - u_{2xx} \\
u_2^*u_{Nx}^* - u_{2x}^*u_N^* & u_2^*u_{Nx} - u_{2x}^*u_N & \cdots & 0 & u_2^*u_{2x} - u_{2x}^*u_2 & u_2^*u_{1x}^* - u_{2x}^*u_1^* & u_2^*u_{1x} - u_{2x}^*u_1 & -4Au_2^* - u_{2xx}^* \\
u_1u_{Nx}^* - u_{1x}u_N^* & u_1u_{Nx} - u_{1x}u_N & \cdots & u_1u_{2x}^* - u_{1x}u_2^* & u_1u_{2x} - u_{1x}u_2 & u_{1x}^*u_1 - u_1^*u_{1x} & 0 & -4Au_1 - u_{1xx} \\
u_1^*u_{Nx}^* - u_{1x}^*u_N^* & u_1^*u_{Nx} - u_{1x}^*u_N & \cdots & u_1^*u_{2x}^* - u_{1x}^*u_2^* & u_1^*u_{2x} - u_{1x}^*u_2 & 0 & u_1^*u_{1x} - u_{1x}^*u_1 & -4Au_1^* - u_{1xx}^* \\
4Au_N^* + u_{Nxx}^* & 4Au_N + u_{Nxx} & \cdots & 4Au_2^* + u_{2xx}^* & 4Au_2 + u_{2xx} & 4Au_1^* + u_{1xx}^* & 4Au_1 + u_{1xx} & 0
\end{pmatrix}
\qquad (14)
$$

where $A = \sum_{n=1}^{N} |u_n|^2$. Hence, equations (11) and (13) constitute an IST scheme for equation (10) (and equation (8) simultaneously).

From the Jost functions, satisfying the boundary conditions and from the time evolution part of the linear eigenvalue problem, we derive the time-dependent scattering data. The Gel'fand–Levitan–Marchenko equation is derived by defining integral kernels using the standard IST procedure.

4. One-soliton solutions

Taking the eigenvalue ζ of single-soliton solution as

$$\zeta = (\xi + \mathrm{i}\eta) \tag{15}$$

we have the single-soliton solution as

$$u_j(x,t) = \frac{2\eta\gamma_j}{\beta}\exp\left\{\mathrm{i}\left[\tfrac{1}{2}\pi - 8\epsilon\xi(\xi^2 - 3\eta^2)t - 2\xi x\right]\right\}\operatorname{sech}\left[8\epsilon\eta(\eta^2 - 3\xi^2)t - 4\eta x + \theta\right] \tag{16}$$

where

$$\begin{aligned} \gamma_j &= \frac{C_{(2N+1)j}(0)}{\alpha'_{(2N+1)(2N+1)}(\zeta^*)} \\ \beta^2 &= \sum_{p=1}^{2N}\left|\frac{C_{(2N+1)p}(0)}{\alpha'_{(2N+1)(2N+1)}(\zeta^*)}\right|^2 \\ \theta &= \ln(\beta/2\eta). \end{aligned} \tag{17}$$

(Cs and αs are scattering data.) Thus, the single-soliton solution for N-CHNLS equations (8) can be written as

$$\begin{aligned} E_j(Z,T) = \frac{2\eta\gamma_j}{\beta}\exp&\left\{\mathrm{i}\left[\tfrac{1}{2}\pi - 8\epsilon\xi(\xi^2 - 3\eta^2)Z - 2\xi\left(T - \frac{Z}{12\epsilon}\right) + \frac{1}{6\epsilon}\left(T - \frac{Z}{18\epsilon}\right)\right]\right\} \\ &\times \operatorname{sech}\left[8\epsilon\eta(\eta^2 - 3\xi^2)Z - 4\eta\left(T - \frac{Z}{12\epsilon}\right) + \theta\right]. \end{aligned} \tag{18}$$

In a similar way, one can generate N-soliton solutions for N-CHNLS equations. The one-soliton solution for one-, two- and three-field (i.e. $j = 1, 2$ and 3 in equation (18)) propagation from the IST scheme is the same as that generated from the Bäcklund transformation in [9, 15].

5. Conclusion

Thus, in this paper, we have considered N-CHNLS equations which govern simultaneous propagation of N fields in a fibre medium with all higher-order effects, namely HOD, self-steepening and stimulated Raman scattering. Using Painlevé analysis, an integrable form of the N-CHNLS equations was derived. Similarly to the single-field propagation case [9], the asymmetrical temporal broadening due to HOD is exactly counterbalanced by the asymmetrical spectral broadening due to Kerr dispersion and stimulated Raman scattering for the condition $\alpha_3{:}\alpha_4{:}(\alpha_4 + \alpha_5) = 1{:}6{:}3$, in N-field propagation also. Then, using suitable variable transformations, N-CHNLS equations have been transformed to N-coupled complex modified KdV-type equations. With the help of a $(2N + 1) \times (2N + 1)$ linear eigenvalue problem, exact one-soliton solutions were generated from the IST scheme. We hope that these analytical results will help in understanding WDM in fibre media with all higher-order effects.

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